

Some remarks on the dyadic Rademacher maximal function

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ABSTRACT. Properties of a maximal function for vector-valued martingales were studied by the author in an earlier paper. Restricting here to the dyadic setting, we prove the equivalence between (weighted) L^p inequalities and weak type estimates, and discuss an extension to the case of locally finite Borel measures on \mathbb{R}^n . In addition, to compensate for the lack of an L^∞ inequality, we derive a suitable BMO estimate. Different dyadic systems in different dimensions are also considered.

1. Introduction

The *Rademacher maximal function* was originally introduced by Hytönen, McIntosh and Portal [6] in order to prove a ‘Carleson’s embedding theorem’ for vector-valued functions. It provides a vector-valued analogue for the standard dyadic maximal function by replacing the suprema of local averages with their R-bounds — a stochastic concept better suited for sets of vectors in infinite-dimensional Banach spaces. L^p inequalities for this new maximal function were found to define a non-trivial Banach space property *RMF*, possessed by most Banach spaces, yet not by all (e.g. ℓ^1).

The author studied this maximal function in a more general setting of martingales [7] and showed, employing somewhat lengthy arguments along the lines of [8] and [2], that the RMF property is characterized by a certain weak type estimate. A significantly simpler approach is available if one restricts considerations to the original setting of dyadic cubes. Doing so enables us to extend the characterization of the RMF property and answer also other natural questions concerning the Rademacher maximal function. Nevertheless, the question whether the RMF property follows from the better known UMD property remains open.

Recently, the Rademacher maximal function has found applications in vector-valued *Tb* theorems, where one is typically led to study paraproduct operators, whose boundedness relies on Carleson’s embedding theorems. This was the case in an earlier version of [4] concerning a (global) vector-valued non-homogeneous *Tb* theorem and in a current version of its local counterpart [5].

The extended characterization of the RMF property is stated in Theorem 1, whereas Theorem 2 entails the BMO estimate. Theorem 3 states the equivalence of L^p inequalities with respect to different dyadic systems and the corresponding result for different dimensions is presented in Theorem 4. The characterization provided by Theorem 1 is discussed in a more general setting of locally finite Borel measures in Section 5.

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R-bounds. Let X be a Banach space and write (ε_k) for a sequence of independent random variables attaining values $+1$ and -1 each with probability $1/2$. Comparison of randomized sums (and their expectations \mathbb{E}) with square sums lies at the heart of our interest.

DEFINITION. A set $S \subset X$ is said to be *R-bounded*¹ if there exists a constant C such that

$$\left(\mathbb{E} \left\| \sum_k \varepsilon_k \lambda_k \xi_k \right\|^2 \right)^{1/2} \leq C \left(\sum_k |\lambda_k|^2 \right)^{1/2}$$

for all (finite) collections of vectors $\{\xi_k\} \subset S$ and scalars $\{\lambda_k\}$. The smallest such C is the R-bound $\mathcal{R}(S)$.

REMARK.

- R-bounds satisfy the following ‘triangle inequality’: For $S, S' \subset X$ one has

$$|\mathcal{R}(S) - \mathcal{R}(S')| \leq \mathcal{R}(S \pm S') \leq \mathcal{R}(S) + \mathcal{R}(S').$$

Furthermore, R-bounds are monotone and subadditive in the sense that

$$\mathcal{R}(S) \leq \sum_m \mathcal{R}(S_m) \quad \text{whenever} \quad S \subset \bigcup_m S_m \subset X.$$

In particular, for any sequence $(\xi_k) \subset X$ one has $\mathcal{R}(\xi_1, \xi_2, \dots) \leq \sum_k \|\xi_k\|$.

- R-bounds always exceed uniform bounds, that is,

$$\sup_{\xi \in S} \|\xi\| \leq \mathcal{R}(S).$$

Moreover, that $\mathcal{R}(S) \lesssim^2 \sup_{\xi \in S} \|\xi\|$ holds for all $S \subset X$ is equivalent with X having *type 2*. Recall that X is said to have type $p \in [1, 2]$ if

$$\left(\mathbb{E} \left\| \sum_k \varepsilon_k \xi_k \right\|^2 \right)^{1/2} \lesssim \left(\sum_k \|\xi_k\|^p \right)^{1/p}$$

for all (finite) collections $\{\xi_k\} \subset X$.

The Rademacher maximal function. Let us consider a system $\mathcal{D} = \bigcup_{k \in \mathbb{Z}} \mathcal{D}_k$ of *dyadic cubes*, where each \mathcal{D}_k partitions \mathbb{R}^n into cubes of sidelength 2^{-k} and every $Q \in \mathcal{D}_k$ is a union of 2^n smaller cubes $R \in \mathcal{D}_{k+1}$. A standard example of such a system is given by $\mathcal{D}_k = \{2^{-k}([0, 1)^n + m) : m \in \mathbb{Z}^n\}$. Note that every $Q \in \mathcal{D}$ is contained in a unique larger cube Q^* with $|Q^*| = 2^n |Q|$ ($|\cdot|$ refers to the Lebesgue measure) and that for any two $Q, R \in \mathcal{D}$ the intersection $Q \cap R$ is either \emptyset , Q or R . By *maximality* of a dyadic cube Q in a given subcollection of \mathcal{D} we mean that there does not exist a cube R in the same subcollection for which $Q \subsetneq R$. Note that maximal cubes are always disjoint and cover the same area as the whole subcollection. Unless otherwise stated, Q and R will always stand for dyadic cubes in a given system.

For $1 \leq p \leq \infty$, we denote by $L^p(X)$ the Lebesgue–Bochner space of p -integrable functions (essentially bounded for $p = \infty$) on \mathbb{R}^n taking values in a Banach space X .

DEFINITION. The *Rademacher maximal function* of an $f \in L^1_{\text{loc}}(X)$ is given by

$$\mathcal{M}f(x) = \mathcal{R}(\langle f \rangle_Q : Q \ni x), \quad x \in \mathbb{R}^n, \quad \text{where} \quad \langle f \rangle_Q = \frac{1}{|Q|} \int_Q f(y) dy.$$

REMARK.

- If X has type 2, then R-bounds are comparable with uniform bounds and so $\mathcal{M}f$ is controlled pointwise by the standard dyadic maximal function

$$Mf(x) = \sup_{Q \ni x} \|\langle f \rangle_Q\|.$$

- Functions with finite Haar decomposition form a dense subspace of $L^p(X)$ for every $p \in [1, \infty)$ and for such f we have $\|\mathcal{M}f\|_{L^p} < \infty$.
- Averages over large cubes have finite R-bounds for any $f \in L^p(X)$ with $1 \leq p < \infty$, that is, given any dyadic cube Q we have $\mathcal{R}(\langle f \rangle_R : R \supset Q) < \infty$.
- \mathcal{M} preserves the dyadic support of functions with zero mean: If $\text{supp } b \subset Q$ and $\int b = 0$, then for every $x \notin Q$ and every $R \ni x$ we have $\langle b \rangle_R = 0$, since either $R \cap Q = \emptyset$ or $R \supset Q$. Consequently, $\mathcal{M}b(x) = 0$ for $x \notin Q$.

¹This coincides with the concept of R-boundedness by Berkson and Gillespie [1], when vectors are viewed as operators from scalars to X .

²By $\alpha \lesssim \beta$ we mean that there exists a constant C such that $\alpha \leq C\beta$. Quantities α and β are comparable, $\alpha \approx \beta$, if $\alpha \lesssim \beta$ and $\beta \lesssim \alpha$.

- If X is a space of operators and has thus an intrinsic concept of R-boundedness it can be used to replace the definition above. This point of view was elaborated in [7, Section 3]. The results of this paper remain valid with this adjustment.

2. L^p inequalities and weak type estimates

In this section we prove that for any Banach space X and any $1 < p < \infty$, the L^p inequality

$$\int_{\mathbb{R}^n} \mathcal{M}f(x)^p dx \lesssim \int_{\mathbb{R}^n} \|f(x)\|^p dx,$$

abbreviated as $\mathcal{M} : L^p(X) \rightarrow L^p$, is equivalent with weak type estimates both on $L^1(X)$ and on the Hardy space $H^1(X)$. Moreover, we consider weighted L^p inequalities for weights in the (dyadic) Muckenhoupt classes A_p .

Weak type estimates. The Hardy space $H^1(X)$ is taken to consist of those $f \in L^1(X)$ for which the dyadic maximal function Mf is integrable, so that the norm $\|f\|_{H^1(X)} := \|Mf\|_{L^1}$ is finite. An equivalent description is given in terms of *atoms*: A function $a \in L^q(X)$, where $1 < q \leq \infty$, is said to be a q -atom if there is a dyadic cube Q so that

$$\text{supp } a \subset Q, \quad \int_Q a(x) dx = 0, \quad \text{and} \quad \|a\|_{L^q(X)} \leq |Q|^{-1/q'},$$

q' being the Hölder conjugate of q . Note that every q -atom a satisfies $\|a\|_{H^1(X)} \lesssim 1$. Now $H^1(X)$ consists of exactly those $f \in L^1(X)$ which admit, for every $q \in (1, \infty]$, a decomposition into q -atoms a_k so that

$$f = \sum_k \lambda_k a_k, \quad \text{with} \quad \sum_k |\lambda_k| < \infty.$$

The weak type Hardy space estimate is the requirement that

$$|\{x \in \mathbb{R}^n : \mathcal{M}f(x) > \lambda\}| \lesssim \frac{1}{\lambda} \|f\|_{H^1(X)}$$

for all $\lambda > 0$. We write this as $\mathcal{M} : H^1(X) \rightarrow L^{1,\infty}$ (and similarly for $L^1(X)$).

The key to the derivation of an L^p inequality from a weak type estimate is a suitable distributional inequality, where \mathcal{M} is controlled by another maximal operator. For $1 \leq q < \infty$ we define

$$M_q f(x) = \sup_{Q \ni x} \left(\frac{1}{|Q|} \int_Q \|f(y)\|^q dy \right)^{1/q}.$$

LEMMA 1. *Suppose that $\mathcal{M} : H^1(X) \rightarrow L^{1,\infty}$ and let $1 < q < \infty$. If f has a finite Haar decomposition and Q is maximal among cubes for which $\mathcal{R}(\langle f \rangle_R : R \supset Q) > \lambda$ for a given $\lambda > 0$, then*

$$|\{x \in Q : \mathcal{M}f(x) > 2\lambda, M_q f(x) \leq \delta\lambda\}| \lesssim \frac{\delta}{1-\delta} |Q|$$

for all $\delta \in (0, 1)$. Consequently, for every $\lambda > 0$ and $\delta \in (0, 1)$, we have

$$|\{x \in \mathbb{R}^n : \mathcal{M}f(x) > 2\lambda, M_q f(x) \leq \delta\lambda\}| \lesssim \frac{\delta}{1-\delta} |\{x \in \mathbb{R}^n : \mathcal{M}f(x) > \lambda\}|.$$

PROOF. Given an f with a finite Haar decomposition and a $\lambda > 0$, let Q be maximal among cubes for which $\mathcal{R}(\langle f \rangle_R : R \supset Q) > \lambda$.

If $\mathcal{M}f(x) > 2\lambda$ for an $x \in Q$, then $\mathcal{R}(\langle f \rangle_R : R \subset Q, R \ni x) > \lambda$, since $\mathcal{R}(\langle f \rangle_R : R \supset Q^*) \leq \lambda$ by maximality of Q . If also $M_q f \leq \delta\lambda$ somewhere in Q , then

$$\begin{aligned} \mathcal{M}(1_Q(f - \langle f \rangle_Q))(x) &= \mathcal{R}(\langle f \rangle_R - \langle f \rangle_Q : R \subset Q, R \ni x) \\ &\geq \mathcal{R}(\langle f \rangle_R : R \subset Q, R \ni x) - \|\langle f \rangle_Q\| \\ &> (1 - \delta)\lambda, \end{aligned}$$

as $\|\langle f \rangle_Q\| \leq M_q f(y)$ for any $y \in Q$.

Now $1_Q(f - \langle f \rangle_Q)$ is q -atom multiplied by $2|Q|^{1/q'} \|1_Q f\|_{L^q(X)}$ and so from $\mathcal{M} : H^1(X) \rightarrow L^{1,\infty}$ it follows that

$$\begin{aligned} |\{x \in Q : \mathcal{M}f(x) > 2\lambda, M_q f(x) \leq \delta\lambda\}| &\leq |\{x \in Q : \mathcal{M}(1_Q(f - \langle f \rangle_Q))(x) > (1 - \delta)\lambda\}| \\ &\lesssim \frac{1}{(1 - \delta)\lambda} \|1_Q(f - \langle f \rangle_Q)\|_{H^1(X)} \\ &\lesssim \frac{1}{(1 - \delta)\lambda} |Q|^{1/q'} \|1_Q f\|_{L^q(X)}. \end{aligned}$$

Assuming that $M_q f \leq \delta\lambda$ somewhere in Q , we obtain

$$\left(\int_Q \|f(x)\|^q dx \right)^{1/q} \leq |Q|^{1/q} \inf_{x \in Q} M_q f(x) \leq |Q|^{1/q} \delta\lambda,$$

so that from $|Q|^{1/q'} |Q|^{1/q} = |Q|$ we arrive at

$$|\{x \in Q : \mathcal{M}f(x) > 2\lambda, M_q f(x) \leq \delta\lambda\}| \lesssim \frac{\delta}{1 - \delta} |Q|.$$

The set $\{x \in \mathbb{R}^n : \mathcal{M}f(x) > \lambda\}$ can of course be decomposed into a disjoint union of maximal cubes in the previous sense and so

$$|\{x \in \mathbb{R}^n : \mathcal{M}f(x) > 2\lambda, M_q f(x) \leq \delta\lambda\}| \lesssim \frac{\delta}{1 - \delta} |\{x \in \mathbb{R}^n : \mathcal{M}f(x) > \lambda\}|$$

for all $\delta \in (0, 1)$. \square

REMARK. From $\mathcal{M} : L^1(X) \rightarrow L^{1,\infty}$ one can deduce a similar distributional inequality for $q = 1$.

Weights. For $1 < p < \infty$, the (dyadic) Muckenhoupt class A_p consists of *weights* w (non-negative and locally integrable) such that

$$\left(\frac{1}{|Q|} \int_Q w(x) dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{1-p'} dx \right)^{p-1} \lesssim 1$$

for every dyadic cube Q . This is equivalent to the requirement that, for any Banach space X , $M_1 : L^p(w; X) \rightarrow L^p(w)$, i.e.

$$\int_{\mathbb{R}^n} M_1 f(x)^p w(x) dx \lesssim \int_{\mathbb{R}^n} \|f(x)\|^p w(x) dx.$$

Due to the ‘reverse Hölder property’ of Muckenhoupt weights (see [3, Chapter IV]), every weight in A_p belongs to a smaller class $A_{p/q}$ for some $q > 1$. Furthermore, every such weight w satisfies the following: There exists a $\gamma > 0$ such that, whenever $E \subset Q$ for a dyadic cube Q , we have

$$(*) \quad \frac{w(E)}{w(Q)} \lesssim \left(\frac{|E|}{|Q|} \right)^\gamma.$$

Here, as usual, w is also used to denote the measure $w(x) dx$.

Characterization of the RMF property. We are now in the position to characterize the RMF property of a Banach space by the equivalent conditions in the following statement:

THEOREM 1. *The following conditions are equivalent for any Banach space X :*

- (i) $\mathcal{M} : L^p(w; X) \rightarrow L^p(w)$ for all $p \in (1, \infty)$ and any $w \in A_p$,
- (ii) $\mathcal{M} : L^p(X) \rightarrow L^p$ for some $p \in (1, \infty)$,
- (iii) $\mathcal{M} : L^1(X) \rightarrow L^{1,\infty}$,
- (iv) $\mathcal{M} : H^1(X) \rightarrow L^{1,\infty}$.

PROOF. As (ii) is a special case of (i), the equivalence is obtained by proving that (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i).

(ii) \Rightarrow (iii): To perform the Calderón–Zygmund decomposition for an $f \in L^1(X)$ at height λ , let \mathcal{C} denote the collection of maximal cubes among dyadic cubes Q for which $1/|Q| \int_Q \|f(x)\| dx > \lambda$. We decompose f into ‘good’ and ‘bad’ parts according to

$$g = 1_{\mathbb{R}^n \setminus \bigcup \mathcal{C}} f + \sum_{Q \in \mathcal{C}} 1_Q \langle f \rangle_Q$$

$$b = f - g = \sum_{Q \in \mathcal{C}} 1_Q (f - \langle f \rangle_Q) = \sum_{Q \in \mathcal{C}} b_Q.$$

A standard argument employing the assumption $\mathcal{M} : L^p(X) \rightarrow L^p$ applies to the good part and gives

$$|\{x \in \mathbb{R}^n : \mathcal{M}g(x) > \lambda/2\}| \lesssim \frac{1}{\lambda} \|f\|_{L^1(X)}.$$

For the bad part we observe that $\mathcal{M}b = 0$ outside $\bigcup \mathcal{C}$. Indeed, if $x \notin \bigcup \mathcal{C}$ and $R \ni x$, then $\langle b_Q \rangle_R = 0$ for all $Q \in \mathcal{C}$ and so $\langle b \rangle_R = 0$. Consequently, also

$$|\{x \in \mathbb{R}^n : \mathcal{M}b(x) > \lambda/2\}| \leq \left| \bigcup \mathcal{C} \right| \leq \frac{1}{\lambda} \|f\|_{L^1(X)}.$$

(iii) \Rightarrow (iv): This is immediate from the fact that $\|\cdot\|_{L^1(X)} \leq \|\cdot\|_{H^1(X)}$.

(iv) \Rightarrow (i): Given a $p \in (1, \infty)$ and a $w \in A_p$, we choose a $q \in (1, p)$ such that $w \in A_{p/q}$. Any f with a finite Haar decomposition will then satisfy, for all $\lambda > 0$ and $\delta \in (0, 1)$, the inequality

$$w(\{x \in \mathbb{R}^n : \mathcal{M}f(x) > 2\lambda, M_q f(x) \leq \delta\lambda\}) \lesssim \left(\frac{\delta}{1-\delta}\right)^\gamma w(\{x \in \mathbb{R}^n : \mathcal{M}f(x) > \lambda\}),$$

with some $\gamma > 0$. Indeed, we may write $\{x \in \mathbb{R}^n : \mathcal{M}f(x) > \lambda\}$ as a disjoint union of dyadic cubes Q that are maximal with respect to $\mathcal{R}(\langle f \rangle_R : R \supset Q) > \lambda$, and then appeal to Lemma 1 and to (*) with $E = \{x \in Q : \mathcal{M}f(x) > 2\lambda, M_q f(x) \leq \delta\lambda\}$ to see that there exists a $\gamma > 0$ so that

$$w(\{x \in Q : \mathcal{M}f(x) > 2\lambda, M_q f(x) \leq \delta\lambda\}) \lesssim \left(\frac{\delta}{1-\delta}\right)^\gamma w(Q)$$

for all $\delta \in (0, 1)$.

Now, writing $\alpha(\delta) = (\delta/(1-\delta))^\gamma$, we obtain

$$\begin{aligned} \|\mathcal{M}f\|_{L^p(w)}^p &= 2^p \int_0^\infty p\lambda^{p-1} w(\{x \in \mathbb{R}^n : \mathcal{M}f(x) > 2\lambda\}) d\lambda \\ &\lesssim 2^p \alpha(\delta) \int_0^\infty p\lambda^{p-1} w(\{x \in \mathbb{R}^n : \mathcal{M}f(x) > \lambda\}) d\lambda \\ &\quad + 2^p \int_0^\infty p\lambda^{p-1} w(\{x \in \mathbb{R}^n : M_q f(x) > \delta\lambda\}) d\lambda \\ &= 2^p \alpha(\delta) \|\mathcal{M}f\|_{L^p(w)}^p + \frac{2^p}{\delta^p} \|M_q f\|_{L^p(w)}^p. \end{aligned}$$

Observing that $M_q f(x)^p = M_1 g(x)^{p/q}$ for the scalar function $g(x) = \|f(x)\|^q$, we may deduce from $w \in A_{p/q}$ that

$$\|M_q f\|_{L^p(w)}^p = \int_{\mathbb{R}^n} M_1 g(x)^{p/q} w(x) dx \leq C_{p,q} \int_{\mathbb{R}^n} |g(x)|^{p/q} w(x) dx = C_{p,q} \|f\|_{L^p(w; X)}^p.$$

Choosing δ small enough so that $\alpha(\delta) < 1/2^p$, we obtain after rearrangement that

$$\|\mathcal{M}f\|_{L^p(w)}^p \lesssim \frac{2^p (C_{p,q})^p}{(1 - 2^p \alpha(\delta)) \delta^p} \|f\|_{L^p(w; X)}^p.$$

□

REMARK.

- Condition (i) can also be seen to follow from (iii) by using a distributional inequality as in Lemma 1, but with $q = 1$.

- From condition (ii) it also follows that $\mathcal{M} : H^1(X) \rightarrow L^1$ as can easily be seen from the action of \mathcal{M} on a p -atom a supported in Q :

$$\int_{\mathbb{R}^n} \|\mathcal{M}a(x)\| dx \leq |Q|^{1/p'} \left(\int_Q \mathcal{M}a(x)^p dx \right)^{1/p} \lesssim |Q|^{1/p'} \left(\int_Q \|a(x)\|^p dx \right)^{1/p} \leq 1.$$

- The UMD property of a Banach space X can be characterized by an analogous result for the *dyadic square function* given by

$$Sf(x) = \left(\mathbb{E} \left\| \sum_{Q,\eta} \varepsilon_{Q,\eta} \langle f, h_{Q,\eta} \rangle h_{Q,\eta}(x) \right\|^2 \right)^{1/2},$$

where $\{h_{Q,\eta} : Q \in \mathcal{D}, \eta = 1, \dots, 2^n - 1\}$ constitutes a Haar basis.

Application to paraproducts. Let us briefly note how the weighted L^p inequalities for \mathcal{M} can be applied to vector-valued paraproducts. We define the *paraproduct operator* Π_b associated to a given $b \in \text{BMO}$ by

$$\Pi_b f = \sum_{Q,\eta} \langle f \rangle_Q \langle b, h_{Q,\eta} \rangle h_{Q,\eta}.$$

A standard argument via Carleson's embedding theorem ([6, Theorem 8.2, Corollary B.1]) gives

$$\left(\mathbb{E} \left\| \sum_{Q,\eta} \varepsilon_{Q,\eta} \langle f \rangle_Q \langle b, h_{Q,\eta} \rangle h_{Q,\eta} \right\|_{L^p(w;X)}^p \right)^{1/p} \lesssim \|b\|_{\text{BMO}} \|\mathcal{M}f\|_{L^p(w)},$$

for $w \in A_p$ and $f \in L^p(w;X)$ with $1 < p < \infty$. Assuming that X has UMD, the left hand side controls $\|\Pi_b f\|_{L^p(w;X)}$. If, in addition, X has RMF, then $\|\mathcal{M}f\|_{L^p(w)} \lesssim \|f\|_{L^p(w;X)}$ according to Theorem 1, which establishes the boundedness of Π_b on $L^p(w;X)$. See [6, Appendix B] for historical remarks.

3. A BMO estimate

In contrast to other, more usual maximal operators (such as M_q), \mathcal{M} does not in general map $L^\infty(X)$ boundedly into L^∞ . Indeed, according to [7, Proposition 4.1] we have:

PROPOSITION. *For any Banach space X , $\mathcal{M} : L^\infty(X) \rightarrow L^\infty$ if and only if X has type 2.*

On the other hand, a linearized version of \mathcal{M} was shown in [6, Proposition 7.1] to map $L^\infty(X)$ into a certain vector-valued BMO space. Recall that by the John–Nirenberg inequality the BMO norm of an $f \in L^1_{\text{loc}}(X)$ can be given by any of the equivalent quantities

$$\|f\|_{\text{BMO}(X)} \approx \sup_Q \left(\frac{1}{|Q|} \int_Q \|f(x) - \langle f \rangle_Q\|^p dx \right)^{1/p}, \quad 1 \leq p < \infty.$$

Moreover, the dyadic averages $\langle g \rangle_Q$ in the BMO norm of a scalar function $g \in L^1_{\text{loc}}$ can be replaced by other scalars c_Q according to the formula

$$\|g\|_{\text{BMO}} \approx \sup_Q \inf_{c_Q} \frac{1}{|Q|} \int_Q |g(x) - c_Q| dx.$$

THEOREM 2. *Suppose that $\mathcal{M} : L^p(X) \rightarrow L^p$ for some $1 < p < \infty$. Then*

$$\|\mathcal{M}f\|_{\text{BMO}} \lesssim \|f\|_{\text{BMO}(X)}$$

for any $f \in L^1_{\text{loc}}(X)$ with $\mathcal{M}f < \infty$ almost everywhere.

PROOF. For every dyadic cube Q and every $x \in Q$ we have

$$\begin{aligned} \mathcal{R}(\langle f \rangle_R : R \ni x) &\leq \mathcal{R}(\langle f \rangle_R - \langle f \rangle_Q + \langle f \rangle_{R'} : R \subset Q, R \ni x, R' \supset Q) \\ &\leq \mathcal{R}(\langle f \rangle_R - \langle f \rangle_Q : R \subset Q, R \ni x) + \mathcal{R}(\langle f \rangle_{R'} : R' \supset Q), \end{aligned}$$

where the first term in the last expression equals $\mathcal{M}(1_Q(f - \langle f \rangle_Q))(x)$. Since $\mathcal{M}f < \infty$ almost everywhere, the constant

$$c_Q = \mathcal{R}(\langle f \rangle_{R'} : R' \supset Q)$$

is finite and so for $x \in Q$,

$$0 \leq \mathcal{M}f(x) - c_Q \leq \mathcal{M}(1_Q(f - \langle f \rangle_Q))(x).$$

Consequently, since $\mathcal{M} : L^p(X) \rightarrow L^p$,

$$\begin{aligned} \frac{1}{|Q|} \int_Q |\mathcal{M}f(x) - c_Q| dx &\leq \frac{1}{|Q|} \int_Q \mathcal{M}(1_Q(f - \langle f \rangle_Q))(x) dx \\ &\leq \left(\frac{1}{|Q|} \int_Q \mathcal{M}(1_Q(f - \langle f \rangle_Q))(x)^p dx \right)^{1/p} \\ &\lesssim \left(\frac{1}{|Q|} \int_Q \|f(x) - \langle f \rangle_Q\|^p dx \right)^{1/p} \lesssim \|f\|_{\text{BMO}(X)}, \end{aligned}$$

as required. \square

4. Different dyadic systems and dimensions

Up until now, we have considered the Rademacher maximal function with respect to a fixed dyadic system on the Euclidean space of fixed dimension. It is shown in this section that the L^p boundedness of \mathcal{M} (as described in Theorem 1) depends neither on the system nor the dimension.

Different dyadic systems. Different dyadic systems on \mathbb{R}^n can be expressed by using a parameter $\beta = (\beta_j) \in (\{0, 1\}^n)^\mathbb{Z}$ according to $\mathcal{D}^\beta = \bigcup_{k \in \mathbb{Z}} \mathcal{D}_k^\beta$, with

$$\mathcal{D}_k^\beta = \{2^{-k}([0, 1]^n + m) + \sum_{j>k} 2^{-j} \beta_j : m \in \mathbb{Z}^n\}.$$

The standard system corresponds to $\beta = 0$ and we refer to it by omitting β altogether. L^p boundedness of the Rademacher maximal operator with respect to \mathcal{D}^β is equivalent with uniform L^p boundedness of the truncated operators defined by

$$\mathcal{M}^{\beta, N} f(x) = \mathcal{R}(A_k^\beta f(x) : k \geq -N), \quad N \in \mathbb{N},$$

where

$$A_k^\beta f = \sum_{Q \in \mathcal{D}_k^\beta} 1_Q \langle f \rangle_Q$$

stands for an averaging operator with respect to \mathcal{D}_k^β . A direct calculation shows that averages with respect to \mathcal{D}_k^β can be obtained from those of the standard system by translations:

$$A_k^\beta = \tau_k^{-1} A_k \tau_k, \quad \text{where} \quad \tau_k f(x) = f(x + \sum_{j>k} 2^{-j} \beta_j).$$

Moreover, large (dyadic) translations commute with averaging so that for $k \geq j$ we have

$$A_k \sigma_j = \sigma_j A_k, \quad \text{where} \quad \sigma_j f(x) = f(x + 2^{-j} \beta_j).$$

Now that $\tau_{k-1} = \sigma_k \tau_k$, we see that actually

$$A_k^\beta = \tau_{-N}^{-1} A_k \tau_{-N} \quad \text{whenever} \quad k \geq -N,$$

and hence

$$\mathcal{M}^{\beta, N} f = \tau_{-N}^{-1} \mathcal{M}^N(\tau_{-N} f).$$

Translations preserve L^p norms and so we have arrived at the following result:

THEOREM 3. *Let $1 < p < \infty$. If the Rademacher maximal operator is L^p bounded with respect to some dyadic system on \mathbb{R}^n , then it is L^p bounded with respect to any dyadic system on \mathbb{R}^n .*

Different dimensions. Let us now consider the Rademacher maximal operator in different dimensions and prove the following result:

THEOREM 4. *Let n be an positive integer and $1 < p < \infty$. The Rademacher maximal operator is L^p bounded on \mathbb{R}^n if and only if it is L^p bounded on \mathbb{R} (or even on $[0, 1]$).*

We restrict our attention to the standard dyadic system on \mathbb{R}^n and note that it divides \mathbb{R}^n into 2^n ‘quadrants’ $\{x \in \mathbb{R}^n : \alpha_j x_j \geq 0\}$, where $\alpha \in \{-1, 1\}^n$, in the sense that every cube in the standard system is contained in one of the (essentially disjoint) quadrants. For L^p boundedness of \mathcal{M} on \mathbb{R}^n , it thus suffices to consider one of these quadrants, say $\{x \in \mathbb{R}^n : x_j \geq 0\}$. By density of functions with bounded support, we may, using a scaling argument, restrict to functions supported in the unit cube $[0, 1]^n$ and consider only averages over cubes contained in $[0, 1]^n$.

Writing $\mathcal{C}^n = \bigcup_{k=0}^{\infty} \mathcal{C}_k^n$, where \mathcal{C}_k^n consists of (standard) dyadic cubes $Q \subset [0, 1]^n$ of sidelength 2^{-k} , we have reduced the question to L^p boundedness of

$$\mathcal{M}f(x) = \mathcal{R}(\langle f \rangle_Q : Q \in \mathcal{C}^n, Q \ni x), \quad x \in [0, 1]^n.$$

To see that \mathcal{M} is L^p bounded on $[0, 1]^n$ if and only if it is L^p bounded on $[0, 1]$ we first note that ‘only if’ is immediate from the fact that functions on $[0, 1]$ can be naturally viewed as functions on $[0, 1]^n$ depending only on the first coordinate. For sufficiency, we provide a way to associate dyadic subcubes of $[0, 1]^n$ with dyadic subintervals of $[0, 1]$ in a suitable manner:

LEMMA 2. *There exists a measure preserving map $\varphi : \mathcal{C}^n \rightarrow \mathcal{C}^1$ which respects the partial order of inclusions in the sense that for all $Q \in \mathcal{C}^n$, we have $\varphi(R) \subset \varphi(Q)$ if and only if $R \subset Q$. Moreover, for every $k \geq 0$, the restriction $\varphi_k : \mathcal{C}_k^n \rightarrow \mathcal{C}_{nk}^1$ is bijective.*

PROOF. Agreeing first that $\varphi([0, 1]^n) = [0, 1]$, we proceed inductively. Namely, if $Q = 2^{-k}([0, 1]^n + m) \in \mathcal{C}_k^n$ is mapped to $\varphi(Q) = 2^{-nk}([0, 1] + l) \in \mathcal{C}_{nk}^1$, then each subcube $R \in \mathcal{C}_{k+1}^n$ of Q is of the form

$$R = 2^{-k-1}([0, 1]^n + 2m + (\delta_1, \dots, \delta_n)), \quad \text{with } \delta_j \in \{0, 1\},$$

and we map it to the interval

$$\varphi(R) = 2^{-n(k+1)}([0, 1] + 2^nl + \delta_1 2^{n-1} + \dots + \delta_n 2^0),$$

which is a subinterval of $\varphi(Q)$. Note that each subinterval $I \in \mathcal{C}_{n(k+1)}^1$ of $\varphi(Q)$ is an image of exactly one subcube $R \in \mathcal{C}_{k+1}^n$ of Q so that each restriction φ_k is bijective. \square

Again, by switching to a truncation of \mathcal{M} , it suffices to consider, for each $N \geq 1$, functions on $[0, 1]^n$ that are constant on cubes of \mathcal{C}_N^n . Every such f , when viewed as a function on cubes of \mathcal{C}_N^n , can be transferred, using Lemma 2, to the function $f \circ \varphi_N^{-1}$ on $[0, 1]$ (which is constant on cubes of \mathcal{C}_{nN}^1). Dyadic averages of $f \circ \varphi_N^{-1}$ include the dyadic averages of f ; for every $Q \in \mathcal{C}_k^n$ with $0 \leq k \leq N$ we have

$$\langle f \rangle_Q = \langle f \circ \varphi_N^{-1} \rangle_{\varphi(Q)}.$$

A calculation shows that the L^p norm of $\mathcal{M}f$ is at most the L^p norm of $\mathcal{M}(f \circ \varphi_N^{-1})$:

$$\begin{aligned} \|\mathcal{M}(f \circ \varphi_N^{-1})\|_{L^p([0, 1])}^p &= \frac{1}{2^{nN}} \sum_{J \in \mathcal{C}_{nN}^1} \mathcal{R}(\langle f \circ \varphi_N^{-1} \rangle_I : I \supset J)^p \\ &\geq \frac{1}{2^{nN}} \sum_{J \in \mathcal{C}_{nN}^1} \mathcal{R}(\langle f \circ \varphi_N^{-1} \rangle_{\varphi(Q)} : \varphi(Q) \supset J)^p \\ &= \frac{1}{2^{nN}} \sum_{R \in \mathcal{C}_N^n} \mathcal{R}(\langle f \circ \varphi_N^{-1} \rangle_{\varphi(Q)} : \varphi(Q) \supset \varphi(R))^p \\ &= \frac{1}{2^{nN}} \sum_{R \in \mathcal{C}_N^n} \mathcal{R}(\langle f \rangle_Q : Q \supset R)^p = \|\mathcal{M}f\|_{L^p([0, 1]^n)}^p. \end{aligned}$$

Since the L^p norms of f and $f \circ \varphi_N^{-1}$ are equal, Theorem 4 follows.

5. More general measures

It was shown in [7, Theorem 5.1] that the RMF property of a Banach space X , as described here by the equivalent conditions in Theorem 1, guarantees the boundedness of the Rademacher maximal operator with respect to any filtration on any σ -finite measure space. It is nevertheless interesting to see that the proof of Theorem 1 is also directly applicable to a more general (possibly non-homogeneous) setting, where \mathbb{R}^n is equipped with a locally finite Borel measure μ . We adjust our maximal functions accordingly by writing

$$\langle f \rangle_Q = \frac{1}{\mu(Q)} \int_Q f(y) d\mu(y), \quad Q \in \mathcal{D},$$

which we agree to be zero if $\mu(Q) = 0$.

The Hardy space $H^1(\mu; X)$ and its atoms are defined analogously with the exception that, due to the possibly non-homogeneous nature of the underlying measure space, it is no longer necessary

for every $f \in H^1(\mu; X)$ to have an atomic decomposition. However, Lemma 1 remains valid for truncated \mathcal{M} when μ replaces the Lebesgue measure and

$$M_q f(x) = \sup_{Q \ni x} \left(\frac{1}{\mu(Q)} \int_Q \|f(y)\|^q d\mu(y) \right)^{1/q}.$$

Truncation allows one to circumvent the problem of possible non-existence of maximal cubes.

As the Calderón–Zygmund decomposition is not applicable for general μ , one might instead appeal to a decomposition of Gundy’s ([9, Chapter IV, Section 2]) in order to pass from an L^p inequality to a weak type estimate (cf. [7, Proposition 6.3]).

With these observations, the proof of Theorem 1 can be adjusted to show the following generalization:

THEOREM 5. *Suppose that μ is a locally finite Borel measure on \mathbb{R}^n . The following conditions are equivalent for any Banach space X :*

- (i) $\mathcal{M} : L^p(\mu; X) \rightarrow L^p(\mu)$ for all $p \in (1, \infty)$,
- (ii) $\mathcal{M} : L^p(\mu; X) \rightarrow L^p(\mu)$ for some $p \in (1, \infty)$,
- (iii) $\mathcal{M} : L^1(\mu; X) \rightarrow L^{1,\infty}(\mu)$,
- (iv) $\mathcal{M} : H^1(\mu; X) \rightarrow L^{1,\infty}(\mu)$.

Let us assume, in addition, that μ is positive and (dyadic) doubling, i.e. that

$$0 < \mu(Q^*) \lesssim \mu(Q) < \infty$$

for all $Q \in \mathcal{Q}$. Then the (dyadic) Muckenhoupt classes of weights can be associated to μ by agreeing that $w \in A_p(\mu)$ with $1 < p < \infty$ if

$$\left(\frac{1}{\mu(Q)} \int_Q w(x) d\mu(x) \right) \left(\frac{1}{\mu(Q)} \int_Q w(x)^{1-p'} d\mu(x) \right)^{p-1} \lesssim 1$$

for every dyadic cube Q , which, again, is equivalent to the requirement that,

$$\int_{\mathbb{R}^n} M_1 f(x)^p w(x) d\mu(x) \lesssim \int_{\mathbb{R}^n} \|f(x)\|^p w(x) d\mu(x),$$

for any Banach space X . Notice that the property $(*)$ remains valid when one replaces the Lebesgue measure by μ .

Under this additional assumption, the four conditions in Theorem 5 are equivalent to

$$\mathcal{M} : L^p(w; X) \rightarrow L^p(w) \text{ for all } p \in (1, \infty) \text{ and any } w \in A_p(\mu).$$

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